

**Instructions:**

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Please do not write under the staple. Make clear what work goes with which problem. Put your name or initials on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils, and your coursepack. You may use any result in the coursepack (whether boxed or an exercise). However, you must cite it, and you may not use it to prove itself (or a portion of itself). Each problem is out of 10 points, 100 points maximum. You have 75 minutes.

Questions 1-5 below refer to the ring  $B$ . Here the “numbers” are the subsets of  $\{x, y\}$ ; there are four numbers altogether. For  $a, b \in B$ , we define multiplication as  $a \times b = a \cap b$ , i.e. the intersection of  $a$  and  $b$ . We define addition as  $a + b = a \Delta b$ . This is the symmetric difference of  $a$  and  $b$ , which you will recall is the set  $\{z : (z \in a \wedge z \notin b) \vee (z \in b \wedge z \notin a)\}$  or if you prefer the set  $(a \cup b) \setminus (a \cap b)$ . You may use without proof basic set theory properties (e.g.,  $\cap$  and  $\Delta$  are each commutative and associative).

1. Write out the addition and multiplication tables for  $B$ . (this proves closure)
2. Find  $0_B$ , find  $1_B$  (if it exists), and determine whether  $B$  is commutative. Justify your answers.
3. Prove that  $B$  is a ring, EXCEPT don't prove distributivity. You may use the results of the previous two problems.
4. Find all units and zero divisors of  $B$ . Determine whether this ring is an integral domain. Determine whether this ring is a field. Determine whether this ring is cancellative.
5. Consider  $B'$ , the subsets of  $\{x\}$ . Note that  $B' \subseteq B$ . Prove or disprove that  $B'$  is a subring of  $B$ .
6. Find, with justification, all the associates of  $[2]$  in  $\mathbb{Z}_{12}$ . How many are there?
7. Suppose that  $R$  is a ring with identity. Prove that this identity is unique.
8. Let  $p \in \mathbb{Z}$  with  $p \geq 2$ . Prove that  $p$  is prime, if and only if,  $\mathbb{Z}_p$  is an integral domain.
9. Let  $R$  be a commutative, cancellative, ring with identity. Prove that every prime is irreducible.
10. Consider the ring  $S = \mathbb{Z}[\sqrt{-5}]$ . Prove that  $3 = 3 + 0\sqrt{-5}$  is irreducible and not prime.